Analytical and numerical optimization methods for solution of non-smooth dynamic impact problems

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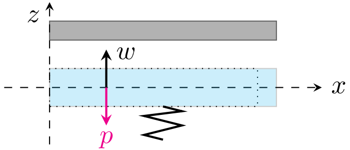
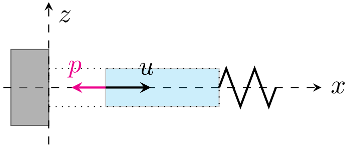
Motivation

- micro (MEMS)/nano (NEMS) electromechanical systems modeled using plates and rods

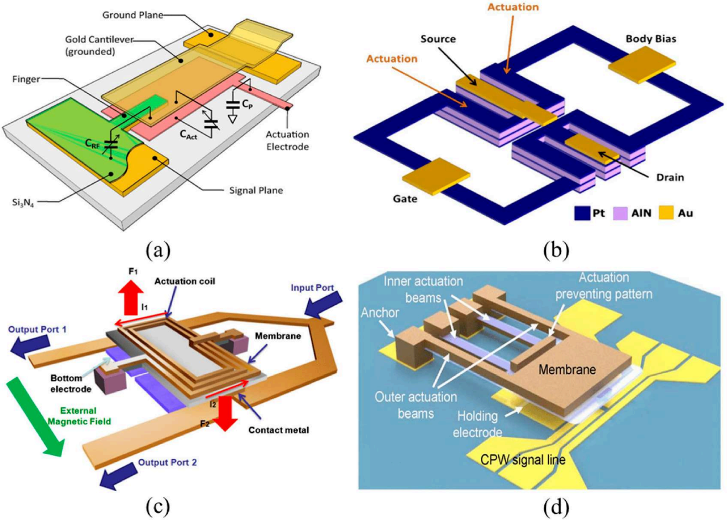


- indentation/ nano-indentation non-destructive testing (NDT) modeled in contact mechanics



- analytical/ Fourier series solution of dynamic impact problems
- numerical solution with no no spurious oscillation, dissipation of energy
- optimal design of flexible structures, distributed sensors and actuators
- inverse problems/ dynamic control for vibration suppression

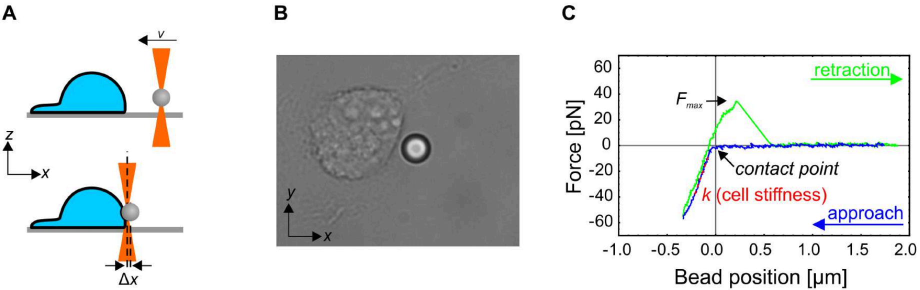
- MEMS-based sensors and actuators



https://www.mdpi.com/2218-273X/15/6/898

Cell indentation

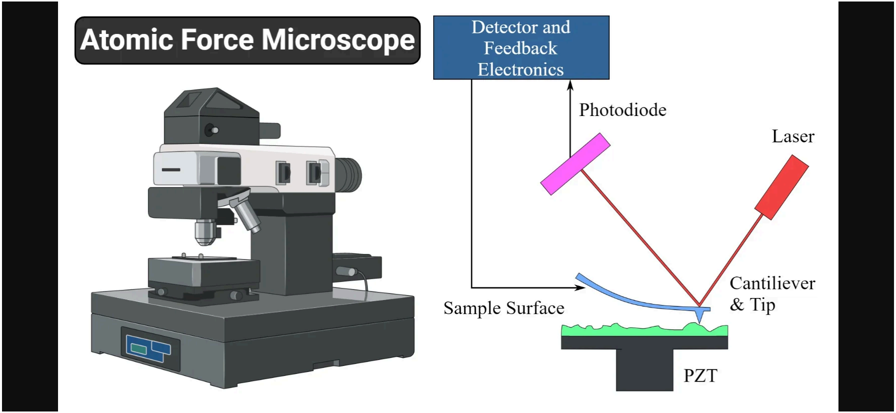
- Cell stiffness measured by lateral indentation



https://journals.plos.org/plosone/article?id=10.1371/journal.pone.0231606

AFM

- Atomic force microscope



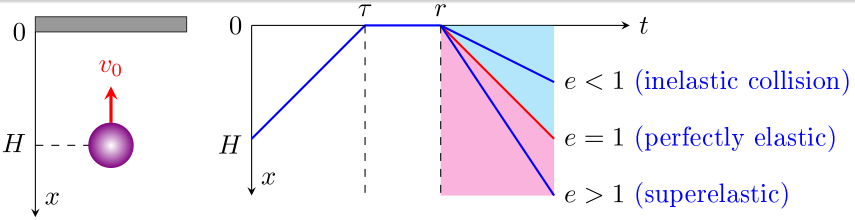
https://microbenotes.com/atomic-force-microscope-afm/

Outline

1. Motivation
2. Impact problems with discontinuous velocity
   - Primal-dual variational formulation
   - Semi-smooth Newton method
   - Space-time (ST) approximation
   - Primal-dual active-set (PDAS) algorithm
3. Analytical and numerical solution of 1D benchmark
   - Elastic bar impact against rigid obstacle
   - Bar bouncing in gravity field
   - Control in the oscillating system for vibration reduction
4. Conclusion/References

Example of multiple solutions

Impact problems with discontinuous velocity



Trajectory $x = u(t)$ of point mass colliding obstacle $x = 0$ at time $\tau = H/v_0$

$$\begin{cases} u_{tt}(t) = \lambda(t), \quad 0 \le u(t) \perp \lambda(t) \le 0 \quad \text{for } t > 0 \\ u(0) = H, \quad u_t(0) = -v_0 \end{cases}$$

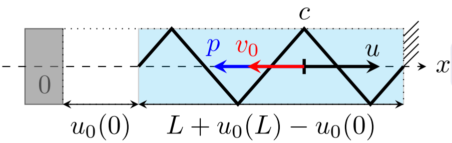
has multiple solutions with arbitrary rebound time $r \ge \tau$ and speed v_1:

$$\begin{cases} u(t) = -v_0(t - \tau), & \lambda(t) = 0 \quad \text{for } t \in (0, \tau) \\ u(t) = 0, & \lambda(t) = 0 \quad \text{for } t \in (\tau, r) \\ u(t) = v_1(t - r), & \lambda(t) = 0 \quad \text{for } t > r \end{cases}$$

the only restitution coefficient $e := \frac{v_1}{v_0} = 1$ at $r = \tau$ conserves kinetic energy

Longitudinal impact problem

Elastic bar of length L and wave speed $c = \sqrt{\frac{E}{\rho}}$ colliding rigid obstacle ($x = 0$)



displacement $u(t,x)$, 1D wave equation

$$u_{tt} - c^2 u_{xx} = -p \text{ for } (t,x) \in Q \qquad (1)$$

in rectangle $Q := (0,T) \times (0,L)$

under axial deformation $\varepsilon = u_x$ due to either/or/both

- initial deformation $u_0(L) - u_0(0) \neq 0$ (Dirichlet)
- initial speed v_0 (Neumann)
- conservative force p

subject to initial and boundary conditions:

$$\begin{cases} u = u_0, \quad u_t = -v_0 \text{ as } t = 0 \\ u = 0 \text{ (Dirichlet) or } u_x = 0 \text{ (Neumann) as } x = L \\ 0 \le u \perp \lambda := c^2 u_x \le 0 \text{ on } \Gamma := \{x = 0\} \end{cases} \qquad (2)$$

Primal-dual variational formulation

- Problem (1)–(2) by Petrov–Galerkin method: find solution in trial space

$$u - u_0 \in V_{0*} := L^2(0,T;H^1(0,L)) \cap \{v \in H^1(0,T;L^2(0,L)),\ v(0,\cdot\,) = 0\}$$

and $\lambda \in L^2(\Gamma)$ validating primal-dual variational equations:

$$\begin{cases} 0 \le u \perp \lambda \le 0 \text{ on } \Gamma \ (u = 0 \text{ as } x = L \text{ for Dirichlet}) \\ \displaystyle\int_Q (-u_t v_t + c^2 u_x v_x + pv)\,dxdt + \int_0^L v_0 v\big|_{t=0}\,dx = -\int_\Gamma \lambda v\,dt \end{cases} \tag{3}$$

for all functions from test space ($v = 0$ as $x = L$ for Dirichlet)

$$v - u \in V_{*0} := L^2(0,T;H^1(0,L)) \cap \{v \in H^1(0,T;L^2(0,L)),\ v(T,\cdot\,) = 0\}$$

lack of regularity: $\lambda \in H^{-1/2}(\Gamma)$

- Reducing λ variational inequality: find $u - u_0 \in V_{0*}$ such that $u \ge 0$ on Γ

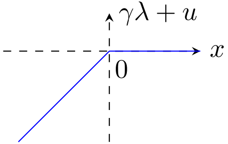
$$\int_Q \big(-u_t(v_t - u_t) + c^2 u_x(v_x - u_x) + p(v-u)\big)\,dxdt + \int_0^L v_0\big(v\big|_{t=0} - u_0\big)\,dx \ge 0$$

for all test functions $v - u \in V_{*0}$ such that $v \ge 0$ on Γ

Semi-smooth Newton method

Complementarity conditions can be expressed by merit function

$$0 \le u \perp \lambda \le 0 \Longleftrightarrow M(z) = 0, \;\; z := (u, \lambda)$$



for arbitrary constant $\gamma > 0$ min-based merit function

$$M(z) := \min(u, -\gamma\lambda) = \min(\gamma\lambda + u, 0) - \gamma\lambda$$

which is non-smooth at $z = 0$

$$\begin{cases} M'_u(z) = 1, & M'_\lambda(z) = 0 & \text{if } \gamma\lambda + u < 0 \\ M'_u(z) = 0, & M'_\lambda(z) = -\gamma & \text{if } \gamma\lambda + u \ge 0 \end{cases}$$

M has the generalized derivative in finite dimensions:

$$\|M(z^\star + h) - M(z^\star) - (M'_u, M'_\lambda)(z^\star + h) \cdot h\|_{\mathbb{R}^N} = \mathcal{O}\big(\|h\|_{\mathbb{R}^N}\big)$$

and in infinite dimensional spaces with gap $1 \le p < q$:[1]

$$\|M(z^\star + h) - M(z^\star) - (M'_u, M'_\lambda)(z^\star + h) \cdot h\|_{L^p} = \mathcal{O}\big(\|h\|_{L^q}\big)$$

[1] M. Hintermüller, K. Ito, K. Kunisch, The primal-dual active set strategy as a semismooth Newton method, *SIAM J. Optimiz.* **13** (2006) 865–888

Space-time (ST) approximation

- Piece-wise polynomials on triangular meshes $\mathcal{T}_h$ in space-time FE space

$$V^h := \left\{ v_h \in C(\overline{Q}),\ v_h \in \mathbb{P}_1(K) \text{ for } K \in \mathcal{T}_h \right\} \subset H^1(Q)$$

its trial and test function sub-spaces ($v_h = 0$ as $x = L$ for Dirichlet):

$$V^h_{0*} := \left\{ v_h \in V^h,\ v_h(0,\cdot\,) = 0 \right\}, \quad V^h_{*0} := \left\{ v_h \in V^h,\ v_h(T,\cdot\,) = 0 \right\}$$

spanned by nodal basis $\{\phi_h\}$ and ϕ_m at $t_m = mh$, $h = \frac{T}{M}$ on $\Gamma = \cup_{m=1}^{M} T_m$

- Approximate (3) using primal-dual active/inactive set formalism

$$\begin{cases} u_h = 0 \text{ on } \mathcal{A}_h := \left\{ \cup T_m :\ \gamma\lambda_m + \frac{1}{2}\big(u_h(t_{m-1},0) + u_h(t_m,0)\big) < 0 \right\} \\ \lambda_h = 0 \text{ on } \mathcal{I}_h := \left\{ \cup \overline{T}_m :\ \gamma\lambda_m + \frac{1}{2}\big(u_h(t_{m-1},0) + u_h(t_m,0)\big) \geq 0 \right\} \\ r_m := \displaystyle\int_Q (-u_{ht}\phi_{mt} + c^2 u_{hx}\phi_{mx} + p\phi_m)\, dxdt + \int_0^L v_0 \phi_m \delta_{m0}\, dx \\ \qquad = \frac{1}{2}(\lambda_m + \lambda_{m+1})h \quad \text{for } m = 0,\dots,M-1 \\ \displaystyle\int_Q (-u_{ht}\phi_{ht} + c^2 u_{hx}\phi_{hx} + p\phi_h)\, dxdt = -\int_0^L v_0 \phi_h\big|_{t=0}\, dx\ \forall \phi_h \neq \phi_m \end{cases} \tag{4}$$

for hyperbolic equations, no M-property, even no P-property!

Space-time primal-dual active-set (ST-PDAS) algorithm

1: Initialize $\mathcal{A}_h^0 = \emptyset$ and $\mathcal{I}_h^0 = [0, T]$; set iteration number $k = 0$

2: Find solution $u_h^k - u_{0h} \in V_{0*}^h$ to the linear system

$$\begin{cases} u_h^k = 0 \quad \text{on } \mathcal{A}_h^k, \quad u_{hx}^k = 0 \quad \text{on } \mathcal{I}_h^k \\ \int_Q (-u_{ht}\phi_{ht} + c^2 u_{hx}\phi_{hx} + p\phi_h)\, dxdt + \int_0^L v_0\phi_h\big|_{t=0}\, dx = 0 \end{cases}$$

for all basis functions $\phi_h \in V_{*0}^h$ with $\phi_h = 0$ on Γ

3: Determine residuals $r_h^k = \{r_m^k\}$ and multipliers $\lambda_h^k = \{\lambda_m^k\}$ from recursion

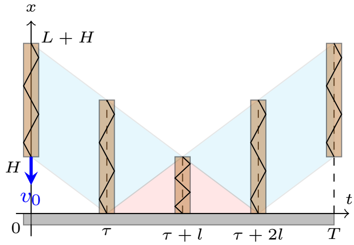
$$\begin{cases} r_m^k = \int_Q (-u_{ht}\phi_{mt} + c^2 u_{mx}\phi_{hx} + g\phi_m)\, dxdt - \int_0^L v_0\phi_m\delta_{m0}\, dx \\ \lambda_0^k := 0, \quad \lambda_{m+1}^k = \frac{2}{h} r_m^k - \lambda_m^k \quad \text{for } m = 0, \dots, M-1 \end{cases}$$

4: Update active/inactive sets

$$\begin{cases} \mathcal{A}_h^{k+1} = \{\cup T_m :\ \gamma\lambda_m^k + \frac{1}{2}\big(u_h^k(t_{m-1}, 0) + u_h^k(t_m, 0)\big) < \mathtt{tol}\}, \\ \mathcal{I}_h^{k+1} = \{\cup \overline{T}_m :\ \gamma\lambda_m^k + \frac{1}{2}\big(u_h^k(t_{m-1}, 0) + u_h^k(t_m, 0)\big) \geq \mathtt{tol}\} \end{cases}$$

5: If $\mathcal{A}_h^{k+1} = \mathcal{A}_h^k$ or cycling then STOP; else set $k = k + 1$ and go to **2**

Analytical and numerical solution of 1D benchmark

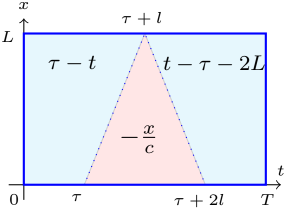


bar initial speed v_0, length L, height H
impact time $\tau := \frac{H}{v_0}$, $l := \frac{L}{c}$, wave speed c

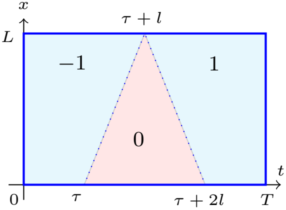
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \quad \text{in } Q = (0,T) \times (0,L) \\ u = H, \quad u_t = -v_0 \quad \text{as } t = 0 \\ u_x = 0 \quad \text{as } x = L \\ 0 \le u \perp \lambda = c^2 u_x \le 0 \quad \text{as } x = 0 \end{cases}$$

bar position $x + u(t,x)$, analytical piece-wise linear solution for $v_0 < c$:

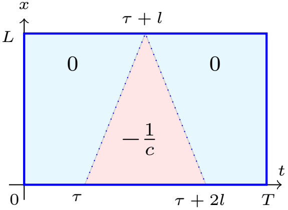
displacement $u(t,x) : v_0$



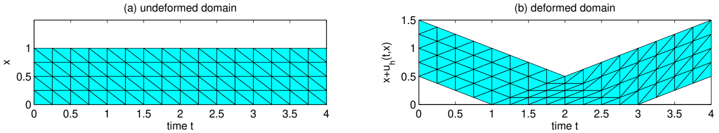
velocity $u_t(t,x) : v_0$



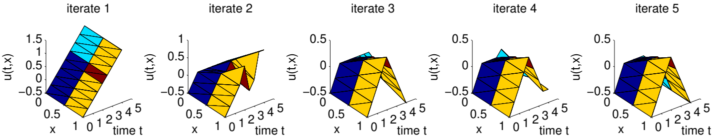
strain $u_x(t,x) : v_0$



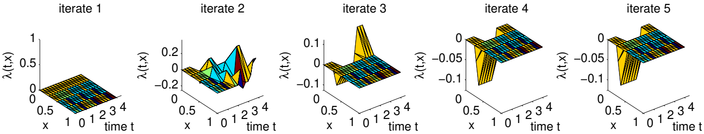
ST-PDAS solution as $c = 1$, $L = 1$, $H = \frac{1}{2}$, $v_0 = \frac{1}{2}$



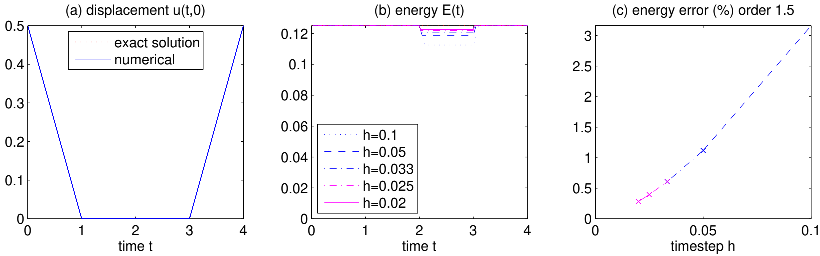
ST-PDAS numerical iterates of displacement $u_h(t,x)$:



contact force $c^2 u_{hx}(t,x)$ implying multiplier λ_h on Γ as $x = 0$:



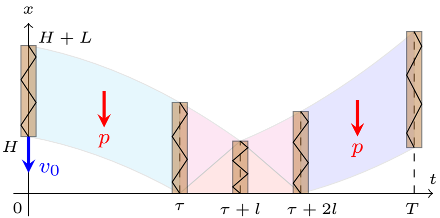
ST-PDAS convergence over uniform triangular mesh



features of ST-PDAS solution:

- algorithm converges in few iterates
- numerical and exact solutions coincide on nodes ($\sim 10^{-14}$)
- solution has no oscillation
- energy has no dissipation
- speed of convergence is super-linear

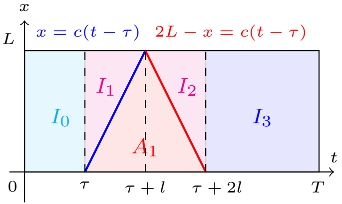
Bar bouncing in gravity field



field p, impact time $\tau := \frac{\sqrt{v_0^2+2pH}-v_0}{p}$

$$\begin{cases} u_{tt}-c^2u_{xx}=-p \quad \text{in } (0,T)\times(0,L) \\ u=H, \quad u_t=-v_0 \quad \text{as } t=0 \\ u_x=0 \quad \text{as } x=L \\ 0\le u\perp\lambda=c^2u_x\le 0 \quad \text{as } x=0 \end{cases}$$

bar position $x+u(t,x)$, generalized D'Alembert formula $u(t,x)=$



if $v_0+p\tau=\sqrt{v_0^2+2pH}\ge 2pl$

$$\begin{cases} H-v_0t-\frac{p}{2}t^2=(\tau-t)\big(v_0+\frac{p}{2}(\tau+t)\big) \text{ in } I_0 \\ (\tau-t)(v_0+p\tau)+p\eta(t,x) \text{ in } I_1 \\ -(v_0+p\tau)\frac{x}{c}+p\eta(t,x) \text{ in } A_1 \\ (v_0+p\tau)(t-\tau-2l)+p\eta(t,x) \text{ in } I_2 \\ (v_0+p\tau)(t-\tau-2l)-\frac{p}{2}(t-\tau-2l)^2 \\ +p\chi(t,x) \text{ in } I_3, \quad \lambda(t)=\begin{cases} -c(v_0+pt) \text{ on } A_1 \\ 0 \text{ else on } \Gamma\setminus A_1 \end{cases} \end{cases}$$

Vibrations

- double Fourier series for clamped bar vibrations of period 4π

$$\eta(t,x) := \frac{2}{Lc^2}\sum_{n=1}^{\infty}\frac{(\cos\nu_n c(t-\tau)-1)\sin\nu_n x}{\nu_n^3}, \quad \nu_n := \frac{\pi}{L}\left(n-\frac{1}{2}\right)$$

$$\begin{cases} \eta_{tt} - c^2\eta_{xx} = -\frac{2}{L}\sum_{n=1}^{\infty}\frac{\sin\nu_n x}{\nu_n} = -1 \text{ as } t\in\mathbb{R},\ x = L \\ \eta = 0, \quad \eta_x = -\frac{2}{Lc^2}\sum_{n=1}^{\infty}\frac{\cos\nu_n c(t-\tau)-1}{\nu_n^2} = \frac{t-\tau}{c} \text{ as } x = 0 \\ \eta = 0, \quad \eta_t = 0 \text{ as } t = \tau; \quad \eta_x = 0 \text{ as } x = L \end{cases}$$

- double Fourier series for free bar vibrations of period 2π

$$\chi(t,x) := \frac{4}{c^2}\sum_{n=1}^{\infty}\frac{\cos\mu_n(c(t-\tau)-2L)\cos\mu_n x}{\mu_n^2} - \frac{2L^2}{3c^2}, \quad \mu_n := \frac{\pi n}{L}$$

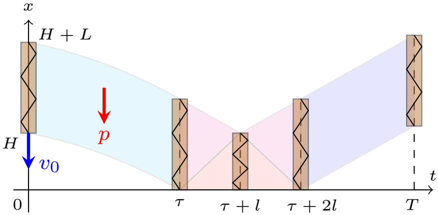
$$\begin{cases} \chi_{tt} - c^2\chi_{xx} = 0 \text{ as } t\in\mathbb{R},\ x\in(0,L) \\ \chi = \frac{x^2-L^2}{c^2}, \quad \chi_t = 0 \text{ as } t = \tau + 2l; \quad \chi_x = 0 \text{ as } x = 0, L \\ \chi = (t-\tau-(2k+1)l)^2 - l^2 \text{ for } t-\tau\in 2l(k,k+1) \text{ as } x = 0 \quad (k\in\mathbb{N}) \end{cases}$$

Control in the oscillating system for vibration reduction

Under step-wise loading

$$p(t) = \begin{cases} p & \text{for } t \in (0, \tau) \\ 0 & \text{for } t \in (\tau, T) \end{cases}$$

non-oscillatory solution to the impact problem given analytically $u(t, x) =$



$$\begin{cases} H - v_0 t - \frac{p}{2} t^2 \text{ in } I_0 \\ (\tau - t)(v_0 + p\tau) \text{ in } I_1 \\ -(v_0 + p\tau)\frac{x}{c} \text{ in } A_1 \\ (v_0 + p\tau)(t - \tau - 2l) \text{ in } I_2 \cup I_3 \end{cases}$$

$c = 1,\ L = \frac{4}{5},\ H = \frac{1}{2},\ p = \frac{1}{8},\ v_0 = \frac{1}{8}$
and $\tau = 2,\ r = \frac{3}{2}$

$$\lambda(t) = \begin{cases} -c(v_0 + p\tau) \text{ on } A_1 \\ 0 \text{ else on } \Gamma \setminus A_1 \end{cases}$$

Conclusion

- Motivation stemming from elastodynamic contact problems with non-smooth velocities, which are stated in a weak variational form
- Implementing by space-time (ST) finite-element approximation and primal-dual active set (PDAS) method of semi-smooth Newton type
- 1D benchmark modeling and constructing explicit solutions for axial impact of an elastic bar against a rigid obstacle
- Numerical ST-PDAS experiments on uniform triangulations yielding high accuracy, no oscillation, no dissipation, super-linear convergence
- Inverse problems to control oscillations and suppress vibrations
- Challenging 2D/3D analysis

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